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Dynamically triangulated random surfaces

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *Dynamically triangulated random surfaces are studied. In view of the cellular automata description of quantum gravity, the covariant formulation is achieved by varying the adjaciency matrix. Computer simulations are discussed.*

INTRODUCTION

The theory described below was supported by Israel Moisseevitch Gelfand few years ago, when nobody believed in it. This proves once again his extraordinary intuition and his open-mindness, which are among many reasons why he contributed so much to the modern mathematics and physics. It is also worth mentioning here that the whole idea of Monte Carlo simulations of the functional integral, was proposed by Gelfand at the very early days of quantum field theory, and it is our fault that we did not appreciate it for such a long time.

The idea of triangulation is as old as geometry. Regge was the first to apply this idea to quantum gravity, by only the lengths of links were treated as dynamical variables in Regge calculus. The triangulation itself, i.e. adjacency matrix was kept fixed according to some optimal, say, regular lattice.

This form of Regge calculus never made the practical computational scheme not

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mentioning the exact solutions of quantum models. The integration over links lengths involved ambiguities, and created tremendous problems from computational point of view.

From purely theoretical point of view the problem was to finde the discrete analogue of general covariance of the gravitation theory. The triangulated model should be formulated in the general covariant form to have the chance to become the quantum gravity in continuum limit.

At it was recently undestood by F. David, and V.A. Kazakov, I.K. Kostov & myself, the covariant formulation can be achieved by varying the adjacency matrix rather then link lengths. For finite number N of points there is finite number of triangulations of given topology and one should simply sum them all with equal weights in functional integral.

This can be viewed as the cellular automata description of quantum gravity, something very simple conceptually and very efficient computationally. Another pleasant feature is the possibility to solve 2d models exactly in some special cases and compare with continuum field theory.

This approach was developed so far only for the simplest case of two dimensional gravity with matter i.e. for the string theory. The dynamical degrees of freedom are represented by the metric field $g_{\alpha\beta}$ and the position field $\vec{X}$ which fields are functions of the point ξ on a sphere S^2 with some number h of handles. The partition function $Z_h(N)$ of our model is given by the following sum over all triangulations $T_h(N)$ of this sphere

$$Z_h(N) = \sum_{T_h(N)} \prod_{i=1}^{N-1} \int \mathrm{d}\vec{X}_i \exp\left(-\sum_{ij} S_{ij}(\vec{X}_i - \vec{X}_j)^2\right)^2$$

$$\sim \int_{\text{Fixed Area}} \mathrm{D}_g \in \mathrm{D}\vec{X} \exp\left(-\int \mathrm{d}^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha \vec{X} \partial_\beta \vec{X}\right). \tag{1.1}$$

Here S_{ij} = (1 for nearest neighbors and 0 otherwise) is the adjacency matrix of the triangulation.

The correspondence between the discrete and continuum notions is as follows.

label $i = 1, \ldots N$	coordinate $\xi \epsilon S^2$
permutations $i \to i'$	diffeomorphisms $\xi \to xi'$
vertices $\vec{X}_i$	embedding $\vec{X}(\xi)$
adjacency matrix S_{ij}	metric sensor $g_{\alpha\beta}$

$$\sum_{ij} \left(\vec{X}_i - \vec{X}_j\right)^2 S_{ij} \qquad \int \mathrm{d}^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha \vec{X} \partial_\beta \vec{X}$$

deficit angle

$$\sum_i \left(2\pi - \sum_j S_{ij}\pi/3\right)$$

integral curvature

$$\int R \mathrm{d}^2\xi \sqrt{g}$$

distance between triangulations

$$\sum_{ij} \left(S_{ij} - S'_{ij}\right)^2 /2$$

functional norm in metric space

$$\int \delta g_{\alpha\beta} \delta g^{\alpha\beta} \mathrm{d}^2\xi \sqrt{g}$$

The last analogy is needed to justify the summation over all triangulations as the definition of the functional integration over metric tensors. The following heuristic argument in favor of this definition can be given. Functional integration over metric tensor parametrized by some local parameters λ_α with the measure

$$\mathrm{D}g = \sqrt{\det_{a,b} \left\| \frac{\delta g}{\delta \lambda_a} \frac{\delta g}{\delta \lambda_a} \right\|} \prod_a \mathrm{d}\lambda_a \tag{1.2}$$

corresponding to the scalar product

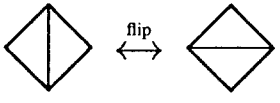
$$\left\| \frac{\delta g}{\delta \lambda_a} \frac{\delta g}{\delta \lambda_a} \right\| = \int \mathrm{d}^2\xi \sqrt{g} \frac{\delta g^{\alpha\beta}}{\delta \lambda_a} \frac{\delta g_{\alpha\beta}}{\delta \lambda_b} \tag{1.3}$$

can be realized as random walk is equidistant steps

$$g \to g' \to g'' \to \ldots \text{with} \|(g - g')^2\| = \|(g' - g'')^2\| = \ldots \tag{1.4}$$

We found the simple unit steps for such walk, covering the whole set of triangulations of given area and given topology. These are flips of links between two adjacent triangles from one to another possible position for fixed positions of all four vertices.

(1.5)



This justified the definition of partition function as sum over triangulations

$$\sum_T = \sum_{\text{Flips } (T_{\text{Reg}})} \to \int \mathrm{D}g \tag{1.6}$$

but of course nobody believed this argument until it was supported by exact calculations.

2. TRIANGULATIONS AS PLANAR CUBIC GRAPHS

The key to all exact solutions of triangulated models in the duality transformation. The dual graph is obtained from the triangulation by connecting the middles of adjacent triangles. By construction this is a planar cubic graph with vertices, lines and loops dual to triangles, lines and vertices of initial triangulation.

Moreover, the coordinates $\vec{X}_i$ of initial vertices can be interpreted as momenta circulating in the corresponding loops of the dual graph. The Gaussian weights of initial links can be reinterpreted as Gaussian propagators of dual links:

$$e^{-(\vec{X}_i-\vec{X}_j)^2} = e^{-P^2_{i*j*}} \tag{2.1}$$

Universality of the critical phenomena suggests that any decreasing propagator with heavy mass would do as well, i.e.

$$(\lambda\phi^3)_{\text{planar}} = \text{2d Gravity} \tag{2.2}$$

The planar graphs were thoroughly studied during the last decade. The heuristic analogy with the random was observed 15 years ago in a pioneer paper by't Hooft, but the internal metric was not yet introduced at that time. Now, with the new interpretation of the planar graphs as dynamical triangulations one may literally apply the planar graph theory to the theory of random surfaces.

The most powerful technique is that of the Brezin, Itzykson, Parisi and Zuber. It reduces to the saddle point calculation of the eigenvalues integral in the large rank matrix representation of the planar graphs. Still it takes some effort to reformulate given triangulated model as the large rank limit of some matrix integral. The known cases will be briefly described below.

3. EXACT SOLUTIONS

- $D = 0$ (Kazakov, David)

This is just the counting of planar graphs:

$$Z_N \sim N^{\gamma_{st}-3} \cdot \exp(CN) \text{ with } \gamma_{st}(D=0) = -\frac{1}{2} \tag{3.1}$$

This value of γ_{st} contradicted the one loop calculations of the quantum Liouville theory, which stimulated the search of the exact solution of this theory (see below).

- $D = -2$ (Kazakov, Kostov, A.M.)

$$\gamma_{st}(D=2) = -1 \text{ and } \langle\vec{X}^2\rangle - \langle\vec{X}\rangle^2 \sim \ln N \tag{3.2}$$

This case required the special combinatorial technique, utilizing the spanning trees representation of the quadratic form in our model. Namely the partition function was reduced to the total number of spanning trees of the cubic graphs, which was calculated by simple combinatorics. The result for the X-variance was obtained by more involved calculations also using the spanning trees.

- $D = 1$ (Kazakov & A.M.)

$$\gamma_{st}(D = 1) = 0(1/\ln N) \text{ and } \langle \vec{X}^2 \rangle - \langle \vec{X} \rangle^2 \sim (\ln N)^2 \tag{3.3}$$

The planar graphs in one dimension were summed in the famous paper of Brezin, Itzykson, Parisi and Zuber but their propagators were $(p^2 + m^2)^{-1}$ rather then Gaussian. Still we interpreted their solution in terms of one dimensional random surface relying on universality. These predictions were also confirmed by exact solutions of the Liouville theory:

- Ising at $D = 0$ (Kazakov & Boulatov)

This case reduces to the two matrix model solved by Mehta. The results for the critical index γ_{st} read

$$\gamma_{st}(\text{Ising}) = -1/3 \tag{3.4}$$

Exact solution of Liouville theory was recently found by Knizhnik, Polyakov, Zamolodchikov. Their results for gamma and X-variance are

$$\gamma_{st}(D) = \frac{D - 1 - \sqrt{(D-1)\,(D-25)}}{12} \tag{3.5}$$

$$\langle \vec{X}^2 \rangle - \langle \vec{X} \rangle^2 \sim C_1 \ln N + C_2 N^{\gamma_{st}} \tag{3.6}$$

Later the Potts and $O(N)\sigma$ models were solved by Kazakov and Kostov. All the critical indices agreed with KPZ solution. This is rather discouraging, since this solution does not make sense at $D > 1$. In case of σ model this corresponds to $N > 2$.

Maybe there are some other models with higher critical dimension or maybe there is the new nontrivial phase at $D > 1$ which is not described by the KPZ solution. The last possibility exists, because the KPZ solution involves the self-consistent ansatz, rather then the most general solution from the first principles.

4. COMPUTER SIMULATIONS

The agreement between the planar graph and the Liouville theories justifies the idea of dynamical triangulations thus opening the way to the numerical experiments with

string theory and quantum gravity. The previous experiments with Regge calculus were unsuccessful partly because of computational difficulties, partly because of conceptual problems with link integration measure.

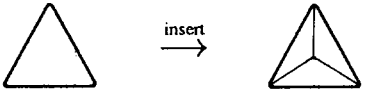
The general algorithm of the Monte Carlo simulations of the dynamically triangulated models consist of three basic steps.

i) adding/ deleting vertices;

ii) updating the triangulation by flips;

iii) updating positions of vertices and/or matter fields.

The simplest way to add vertex to triangulation is to insert the point in the middle of triangle.

(4.1) 

The inverse operation, however, is not always possible, since there may be no vertex with only three neighbors at the general triangulation.

Still this algorithm was successfully used by Boulatov, Kazakov & myself to grow the triangulation starting from tetrahedron. This growing was necessary to keep away from the metastable states, which tend to arise in the cold start simulations (starting from the large regular lattice.) Our colleagues: F. David & A. Billoire and A. Krzywichi et al. in Paris and J. Ambjorn et al. in Copenhagen encountered the similar problems of metastable states and applied the same treatment.

The microcanonical ensemble simulations (i.e. with fixed large number of points) are the simplest of all, as they do not require any fine tuning of parameters. In this case one grows the triangulation thermalizing the system at each number of points and measuring the expectation values and/or correlation functions at some values of number of points, say at power of 2.

The grand canonical ensemble simulations (i.e. with fixed cosmological constant instead of the fixed number of points) runs into various problems. First of all, the fine tuning of the cosmological constant is not enough to increase the dominant number of points. This is because of the decreasing preexponential factor $N^{\gamma_{st}-3}$ in the partition function. The planar graph expansion converges even at the critical value of the cosmological constant.

One may overcome this difficulty by introducing the lower bound for the number of points. The smarter method was recently proposed by Boulatov & Kazakov, who expressed the ratio $\frac{Z_h(N)}{Z_h(N+1)}$ in terms of the microcanonical ensemble averages using

the detailed balance relations. They performed the simulations and observed that $\gamma_{st} = 0$ at $h = 0$ and $D \geq 1$ within small statistical errors, which is so far the only reliable measurement above $D = 1$.

The more general transformation, increasing the number of points, is the link splitting: two links sharing the vertex are transformed into two triangles sharing the link. The original vertex is split into two vertices connected by this new link. The inverse operation shrinks the link to a point, thereby joining two vertices and replacing two triangles by two links.

(4.2) 

In principle, one may construct any triangulation splitting and joining links. The flip can be also represented as split followed by joining. However, inserting points and making flips is usually more efficient, since the changes in triangulation are minimal in this case. Splitting the vertex with large number of neighbors may lead to large variation of the action, lowering the accenptance rate of the algorithm.

The updating of X-coordinates is straighforward in principle, since the probability distributions is Gaussian but in practice this causes some troubles. The simple heat bath algorithm, i.e. updating the randomly chosen point $\vec{X}_i$ according to its probability distribution at fixed other points

$$\exp\left(-\sum_j S_{ij}(\vec{X}_i - \vec{X}_j)^2\right) \tag{4.3}$$

suffers from critical slowing down: it takes large time $O(N^2)$ to update coordinates of the large lattice. There are the acceleration algorithms available such as overrelaxation algorithm or algebraic multigrid method, but they are not easily vectorizable for the Cray type machines.

All these complications led to significant errors in Monte Carlo simulations performed so far event at Cray XMP machines. The worst are the systematic errors coming from the finite number of points (so far, less then one thousand). This is barely enough to build the tree with few cylindrical branches, since the cylinder cannot be made from less then 12 triangles and gluing cylinders at the tree takes about the same.

Therefore, the present Monte Carlo simulations cannot adequately describe the branched polymer phase of the triangulated surface, where such trees play the dominant role.

The significant improvement can be achieved at the Connection Machine 2 or similar massively parallel machines. The 64,000 processors of CM2 can simultaneously update the same amount of vertices and/or links. The parallel code was written recently by L. Jacobs, J. Richardson & myself and is running presently at CM2 in Boston. The first results at $D = 0$ agree with exact solution. The results at $D > 1$ are expected to come in a few months.

The results of the previous simulations at the serial computers at $D > 1$ indicate the scaling laws with nontrivial critical indices such as the fractal dimension of the random surface. These indices vary with the space dimension but which is worse, they also seem to vary with the «irrelevant» parameter α in front of the R^2 term added to the action.

There may be two explanations of this. Either there is the new scaling phase, missed by the Liouville theory, or, which seems more likely to me now, there is an unphysical branched polymer phase with large trees, invisible at small N. In the latter case the scaling laws are finite size artifacts. The only way to check them is to perform computer simulations at much larger scale.

Even thought the new physical phenomena may not be discovered in these simulations, they are necessary to develop the tools to use in the more realistic models. The simplest generalization which do not require new algorithms is the dynamic topology.

The handle can be made at the surface by adding the link between the points ij which were not the nearest neighbors. Or one may make the wormhole by identifying two different triangles.

The problem is the well known instability of the bosonic string against foamy phase transition. The number of triangulations without restriction at the number of handles grows faster then exponential of N, since this is just the number of all ϕ^3 graphs.

So, the entropy of triangulation cannot be compensed by energy with any cosmological constant, in other words, the sum of ϕ^3 graphs diverges at any coupling constant. One may study the foamy phase transition by Boulatov-Kazakov method applied to the ratios $Z_h(N)/Z_{h+1}(N)$.

5. DISCUSSION

The dynamical triangulations so far did not help to study real world in particle theory (in fluid dynamics, though, they are quite useful for the interface simulation). Nevertheless, this is an exciting field and it is quite promising.

First of all, who knows, maybe our world is discrete after all! It is certainly worth trying to study discrete models of quantum gravity from purely theoretical point of view. One may find among these models the new fundamental theories of everything.

Secondly, the reformulation of conventional quantum gravity in completely logical rather then numerical terms promises fantastic efficiency in simulation. The cellular

automata are expected to the many orders of magnitude more powerful computational devices then the modern computers.

The representation of the triangulation as the sequence of flips, and the similar representations for higher dimensional triangulations, provide the basis for the special purpose Gravity Machine, simulating zillions of cells simultaneously.

Thirdly, the nonperturbative phenomena in string theory, such as the space compactification, can be simulated by means of triangulated models. The realistic models require the supersymmetric generalization of the triangulation, which is a challenge for the string theorists. If the triangulation is the discrete graviton, what is the discrete gravitino?

All these ambitious goals are far away, but at least now we may dream of them. We may apply all the experience gained with Monte Carlo simulations in lattice theories to simulations of dynamical triangulations. This may lead to interesting new discoveries, which is certainly the better use of computers, then calculating new digits in old numbers.

Added in proofs (Nov. 1989)

During the last two years, some progress was made both in theory and computer simulations. The multicritical points in random matrix models were studied by Kazakov (V. Kazakov, NBI-HE-89-25, 1989), who made an interesting conjecture that they describe gravity with matter.

The computer simulations were performed at $d = 0$ for 16 thousand triangles using parallel flips at Connection Machine (M. Agishtein, L. Jacobs, A.A. Migdal, and J. Richardson, to be published). The results confirm earlier estimates, in addition new interesting phenomena, such as anomalous diffusion on random graph were observed. The simulations for $d \leq 1$ are in progress. The higher genus surfaces were enumerated at $d = -2$ by Kostov and Mehta, (I. Kostov and M. Mehta, *Phys. Lett.* 189, 118, (1987)), after which a few groups (Brezin and Kazakov, Shenker and Douglas, Gross and Midgal, October 1989, to be published) did the same for $d = 0$, where the exciting nonperturbative phenomena occurred. The general correlation functions on the sphere, and the intriguing relation between Quantum Gravity with matter and half integral power of the Shrödreger operator were found by Gross and myself.

Incidentally, the mathematical theory of half-integral powersof the Shrödreger operator was so beautiful that I am not surprised that Israel Moisseevitch Gelfand contributed to this theory. (I. Gelfand and L. Dikii, *Usp. Matem. Nauk* 30, 5 (1975)).

REFERENCES

F. DAVID: *Nucl. Phys.*, **B257**, 45, 1985.

V.A. KAZAKOV: *Phys. Lett.* **150B**,282, 1985.

J. AMBJØRN, B. DURHUUS and J. FRÖHLICH: *Nucl. Phys.* **B257**, 433, 1985.

M. BANDER and C. ITZYKSON: *Lecture notes in Physics*, Vol. 226 (Springer, Berlin, 1985).

A. JEVICKI and M. NINOMIYA: *Phys. Lett.*, **150 B**, 115, 1985.

V. A. KAZAKOV, I.K. KOSTOV and A.A. MIGDAL: *Phys. Lett*, **157B**, 295, 1985.

F. DAVID: *Nucl. Phys.*, **B257**, [FS14], 543, 1985.

D.V. BOULATOV, V.A. KAZAKOV, I.K. KOSTOV and A.A. MIDGAL: *Nucl. Phys.*, **B275** [FS17], 543, 1986.

I.K. KOSTOV and M.L. METHA: *Phys. Lett.*, **189B**, 118, 1987.

F. DAVID: *Phys. Lett.*, **159B**, 303, 1985.

V.A. KAZAKOV: *Nucl. Phys.*, **B4**, (Proc. Suppl.), 93, 1988.

A. BILLOIRE and F. DAVID: *Phys. Lett.*, **B168**, 279, 1986; *Nucl. Phys.*, **B275**, 617, 1986.

J. JURKIEVIC, A. KRZYWICHI and B. PETERSON: *Phys. Lett.*, **168B**, 273, 1986; **B177**, 89, 1986.

J. AMBJØRN, B. DURHUUS, J. FRÖLICH and P. ORLAND: *Nucl. Phys.*, **B270** [FS16], 457, 1986.

A.A. MIGDAL and V.A. ZMUSHKO: *Serpukhov*, preprint 86-90, 1986.

D.V. BOULATOV and V.A. KAZAKOV: *Phys. Lett.*, **184B**, 247, 1987.

J. AMBJØRN, B. DURHUUS and J. FRÖLICH: *Nucl. Phys.*, **B275** [FS17], 161, 1986.

J. AMBJØRN, PH. DE. FORCRAND, F. KOUKIOU and D. PETRITIS: *Phys. Lett.*, **197B**, 548, 1987.

F. DAVID, J. JURKIEWICZ, A. KRZYWICHI and B. PETERSSON: *Nucl. Phys.*, **B290**, [FS20], 218, 1987.

D.V. BOULATOV and V.A. KAZAKOV: *Niels Bohr Inst.*, preprint 1988.

V.A. KAZAKOV: *Pis'ma Zh. Exp. Teor. Fiz.*, **44**, 105, 1986.

V.A. KAZAKOV: *Phys. Lett.*, **119A**, 140, 1986.

M.A. BERSHADSKI, I.D. VAISBURD and A.A. MIGDAL: *Pis'ma Zh. Exp. Teor. Fiz.*, **43**, 153, 1986.

M.A. BERSHADSKI and .A. MIGDAL: *Phys. Lett.*, **174B**, 393, 1986.

D.V. BOULATOV and V.A. KAZAKOV: *Phys. Lett.*, **184B**, 247, 1987.

V.A. KAZAKOV: *Niels Bohr. Inst.*, preprint 1988.

E. BRÉZIN, C. ITZYKSON, G. PARISI and J.B. ZUBER: *Comm. Math. Phys.*, **59**, 35, 1978.

A.M. POLYAKOV: *Phys. Lett.*, **103B**, 207, 1981.

A.M. POLYAKOV: *Mod. Phys. Lett.*, **A2**, 899, 1987.

V.G. KNIZHNIK, A.M. POLYAKOV and A.B. ZAMOLODCHIKOV: *Mod. Phys. Lett.* **A**, in print, 1988.

A.B. ZAMOLODCHIKOV: *Phys. Lett.*, **117B**, 97, 1982.

C. ITZYKSON and J.B. ZUBER: *J. Math. Phys.*, **21**, 411, 1980.

Vl. S. DOTSENKO and V.A. FATEEV: *Nucl. Phys.*, **B240**, 312, 1984.

M.L. METHA: *Comm. Math. Phys.*, **79**, 327, 1981.

I.K. KOSTOV and M.L. METHA: *Phys. Lett.*, **189B**, 118, 1987.

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